THEORETICAL ASPECTS OF THREE-ASSET PORTFOLIO MANAGEMENT

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ABSTRACT: *The paper deals with three-asset portfolio. It focuses on ordinary investor, for whom the Markowitz's theory of selection of optimal portfolio is often too difficult to use in practise. In the paper, new formulas for calculation of the weights of the assets in three-asset portfolio optimised according to the risk measured by standard deviation are being derived. The paper also deals with comparison of optimisation of two- and three-asset portfolio. Also the formulas for calculation of weights of assets in three-asset portfolio, which is optimised under the pre-defined rate of return, are derived in the paper.*

KEY WORDS: *portfolio, portfolio management, optimisation, portfolio's analysis, standard deviation.*
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INTRODUCTION

Basic aim of every investor is profit. However, profit maximization is not so easy aim to achieve, as every investor has certain level of risk aversion at the same time. Of course, the level of aversion is different for each investor. Profit and risk are two aspects that cannot be separated and between which indirect proportion exists. The higher yield investor wants to gain, the higher risk has to be undertaken. Regular investor prefers the highest yield and the lowest risk at the same time.

Theory of portfolio management deals with these questions. Risk of portfolio can be lowered through diversification, e. i. splitting the investment to several assets. Not all assets can be, however, used for diversification.

It is interesting to observe, which assets are suitable for effective diversification, i. e. diversification, which leads to lowering the whole risk of portfolio. It is also interesting to know, how to create a portfolio with the minimum risk level and with the minimum risk level under the condition of required rate of return.

The aim of this paper is to give answers to the questions stated above in case of three assets.

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PORTFOLIO WITH THE MINIMUM RISK LEVEL

Let us suppose that there is possibility to invest our financial capital to three different assets X_1, X_2, X_3 with expected rates of return $\overline{r_1}, \overline{r_2}, \overline{r_3}$, standard deviations σ_1, σ_2, σ_3 and their covariance's k_{12}, k_{13}, k_{23}. If weights of these assets in the portfolio created will be p_1, p_2, p_3, then variance of such created portfolio will be calculated from the formula

$$\sigma_p^2 = (p_1, p_2, p_3)\begin{pmatrix} \sigma_1^2, & k_{12}, & k_{13} \\ k_{12}, & \sigma_2^2, & k_{23} \\ k_{13}, & k_{23}, & \sigma_3^2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = p_1^2\sigma_1^2 + p_2^2\sigma_2^2 + p_3^2\sigma_3^2 + 2p_1p_2k_{12} + 2p_1p_3k_{13} + 2p_2p_3k_{23}$$

. (1)

As $p_3 = 1 - p_1 - p_2$, then

$$\sigma_p^2 = p_1^2\sigma_1^2 + p_2^2\sigma_2^2 + (1 - p_1 - p_2)^2\sigma_3^2 + 2p_1p_2k_{12} + 2p_1(1 - p_1 - p_2)k_{13} + 2p_2(1 - p_1 - p_2)k_{23}$$

(2)

Theoretically, it is possible to create an infinite number of portfolios from three different assets, each with different weights of assets. Our task is to find the one with the minimum risk level, so weights of assets must meet these equations

$$\frac{\partial\sigma_p^2}{\partial p_1} = 2p_1\sigma_1^2 - 2(1 - p_1 - p_2)\sigma_3^2 + 2p_2k_{12} + 2(1 - p_1 - p_2)k_{13} - 2p_1k_{13} - 2p_2k_{23} = 0,$$

$$\frac{\partial\sigma_p^2}{\partial p_2} = 2p_2\sigma_2^2 - 2(1 - p_1 - p_2)\sigma_3^2 + 2p_1k_{12} - 2p_1k_{13} + 2(1 - p_1 - p_2)k_{23} - 2p_2k_{23} = 0,$$

from which we get

$$p_1\left(\sigma_1^2 + \sigma_3^2 - 2k_{13}\right) + p_2\left(\sigma_3^2 + k_{12} - k_{13} - k_{23}\right) = \sigma_3^2 - k_{13},$$

3)

$$p_1\left(\sigma_3^2 + k_{12} - k_{13} - k_{23}\right) + p_2\left(\sigma_2^2 + \sigma_3^2 - 2k_{23}\right) = \sigma_3^2 - k_{23}.$$

As

$$\frac{\partial\sigma_p^2}{\partial p_1^2} = 2\sigma_1^2 + 2\sigma_3^2 - 4k_{13} > 2(\sigma_1 - \sigma_3)^2 > 0$$

$$\frac{\partial^2\sigma_p^2}{\partial p_2^2} = 2\sigma_2^2 + 2\sigma_3^2 - 4k_{23},$$

$$\frac{\partial^2\sigma_p^2}{\partial p_1\partial p_2} = 2\sigma_3^2 + 2k_{12} - 2k_{13} - 2k_{23},$$

portfolio with weights which meet the system of equations (2) will have the minimum risk level if

$$D=\begin{vmatrix}\sigma_1^2+\sigma_3^2-2k_{13} & , \sigma_3^2+k_{12}-k_{13}-k_{23}\\ \sigma_3^2+k_{12}-k_{13}-k_{23}, & \sigma_2^2+\sigma_3^2-2k_{23}\end{vmatrix}=\left(\sigma_1^2+\sigma_3^2-2k_{13}\right)\left(\sigma_2^2+\sigma_3^2-2k_{23}\right)-$$
$$-\left(\sigma_3^2+k_{12}-k_{13}-k_{23}\right)^2>0 \quad .$$

Moreover, $p_1 > 0$ and $p_2 > 0$ must be valid.
It can be proved, that if

$$k_{ij}\le \min\left\{\sigma_i^2,\sigma_j^2\right\} \tag{4}$$

and if we order the assets so that

$$k_{12}\le k_{13}\le k_{23} \tag{5}$$

then all conditions stated above are met.

If system (3) is solved using Cramer's rule, then after modification we get the formulas for weights of assets in portfolio with the minimum risk level as these

$$p_1=\frac{\left(\sigma_3^2-k_{13}\right)\left(\sigma_2^2-k_{23}\right)-\left(\sigma_3^2-k_{23}\right)\left(k_{12}-k_{23}\right)}{\left(\sigma_1^2+\sigma_3^2-2k_{13}\right)\left(\sigma_2^2+\sigma_3^2-2k_{23}\right)-\left(\sigma_3^2+k_{12}-k_{13}-k_{23}\right)^2}\ ,$$

(6)

$$p_2=\frac{\left(\sigma_3^2-k_{23}\right)\left(\sigma_1^2-k_{13}\right)-\left(\sigma_3^2-k_{13}\right)\left(k_{12}-k_{13}\right)}{\left(\sigma_1^2+\sigma_3^2-2k_{13}\right)\left(\sigma_2^2+\sigma_3^2-2k_{23}\right)-\left(\sigma_3^2+k_{12}-k_{13}-k_{23}\right)^2}\ ,$$

$$p_3=1-p_1-p_2\ .$$

We can formulate these statements:

1. If conditions (4) are met, i. e. $k_{ij}\le \min\left\{\sigma_i^2,\sigma_j^2\right\}$, then assets are suitable for diversification..
2. In such case, we order the assets so that $k_{12}\le k_{13}\le k_{23}$ i. e. so that relation (5) was valid.
3. The weights of assets in the portfolio will be calculated using formulas (6).
4. Expected rate of return of such created portfolio can be calculated from the relation $\overline{r_p}=p_1\overline{r_1}+p_2\overline{r_2}+p_3\overline{r_3}$ and its standard deviation from the formula (1).
5. Such created portfolio has the minimum standard deviation from all the possible portfolios that can be created using assets X_1, X_2, X_3.

Example 1. Let us suppose that there is a possibility to invest money to three different assets with the expected rates of return $\overline{r_1} = 15\%$, $\overline{r_2} = 19\%$, $\overline{r_3} = 16\%$, standard deviations $\sigma_1 = 22\%$, $\sigma_2 = 44\%$, $\sigma_3 = 26\%$ and covariances k_{12} = 75, k_{13} = 130, k_{23} = 190. Find the portfolio with the minimum risk level.

Solution. As conditions (4) are met, assets are suitable for diversification. The relationship (5) is also valid, so the weights can be calculated using formulas (6) as follows

$$p_1 = \frac{(676-130)(1936-190)-(676-190)(75-190)}{(484+676-260)(1936+676-380)-(676+75-130-190)^2} = 0{,}554 \quad ,$$

$$p_2 = \frac{(676-190)(484-130)-(676-130)(75-130)}{(484+676-260)(1936+676-380)-(676+75-130-190)^2} = 0{,}111 \quad ,$$

$$p_3 = 1-0{,}554-0{,}111 = 0{,}335.$$

The standard deviation of this portfolio is

$$\sigma_p = \sqrt{148{,}547+23{,}853+75{,}864+9{,}224+48{,}253+14{,}130} = \sqrt{319{,}871} = 17{,}88$$

and expected rate of return is

$$\overline{r_p} = 0{,}554 \cdot 15 + 0{,}111 \cdot 19 + 0{,}335 \cdot 16 = 15{,}8 \quad .$$

COMPARISON OF INVESTMENT TO THREE-ASSET PORTFOLIO WITH TWO-ASSET PORTFOLIO

If assets are suitable for diversification, then three-asset portfolio with minimum rate of risk is more effective than two-asset portfolio created from any two chosen assets. Šoltés, M. and Šoltés, V. (2003) proved that the weights of two-asset portfolio with minimum risk level can be calculated using formulas $p_1 = \dfrac{\sigma_2^2 - k_{12}}{\sigma_1^2 + \sigma_2^2 - 2k_{12}}$, $p_2 = 1 - p_1$ and standard deviation of this portfolio from the formula $\sigma_p = \sqrt{\dfrac{\sigma_1^2\sigma_2^2 - k_{12}^2}{\sigma_1^2 + \sigma_2^2 - 2k_{12}}}$.

Using assets X_1 and X_2 from example 1 we get portfolio with weights and standard deviation $p_1 = 0{,}82$, $p_2 = 0{,}18$, $\sigma_p = 20{,}26$ a $\overline{r_p} = 15{,}72$.

Using assets X_1 and X_3 we get portfolio with weights and standard deviation $p_1 = 0{,}61$, $p_2 = 0{,}39$, $\sigma_p = 18{,}57$ a $\overline{r_p} = 15{,}39$.

Using assets X_2 and X_3 we get portfolio with weights and standard deviation $p_1 = 0{,}22$, $p_2 = 0{,}78$, $\sigma_p = 23{,}8$ a $\overline{r_p} = 16{,}66$.

As we can see, diversification to two-asset portfolio is not as effective as to three-asset portfolio.

PORTFOLIO WITH THE MINIMUM RISK LEVEL AT THE REQUIRED RATE OF RETURN

Let us consider three assets X_1, X_2, X_3 with expected rates of return $\overline{r_1} < \overline{r_2} < \overline{r_3}$, standard deviations σ_1 , σ_2 , σ_3 and covariance k_{12}, k_{13}, k_{23}, while $k_{ij} \le \min\{\sigma_i^2, \sigma_j^2\}$. We have proved that these assets are suitable for diversification. Let us find the portfolio created using the assets X_1, X_2, X_3 with required rate of return $\overline{r_p} \in (\overline{r_1}, \overline{r_2})$, which has the minimum standard deviation σ_p.

We have $\overline{r_p} = p_1\overline{r_1} + p_2\overline{r_2} + p_3\overline{r_3}$ a $p_1 + p_2 + p_3 = 1$. Solving this system of two equations with three unknown quantities we have

$$p_2 = \frac{\overline{r_3} - \overline{r_p} - (\overline{r_3} - \overline{r_1})p_1}{\overline{r_3} - \overline{r_2}} \quad ,$$

(7)

$$p_3 = \frac{\overline{r_p} - \overline{r_2} + (\overline{r_2} - \overline{r_1})p_1}{\overline{r_3} - \overline{r_2}} \quad .$$

As $p_1>0$, $p_2>0$, $p_3>0$, if p_1 will be stated arbitrary under condition of $\max\left\{0, \frac{\overline{r_2} - \overline{r_p}}{\overline{r_2} - \overline{r_1}}\right\} < p_1 < \frac{\overline{r_3} - \overline{r_p}}{\overline{r_3} - \overline{r_1}}$, then such portfolios will have required rate of return $\overline{r_p}$, but different standard deviations.

Let us mark

$$a = \frac{\overline{r_3} - \overline{r_p}}{\overline{r_3} - \overline{r_2}} \ , \qquad b = \frac{\overline{r_3} - \overline{r_1}}{\overline{r_3} - \overline{r_2}} \ , \qquad c = \frac{\overline{r_p} - \overline{r_2}}{\overline{r_3} - \overline{r_2}} \ , \qquad d = \frac{\overline{r_2} - \overline{r_1}}{\overline{r_3} - \overline{r_2}} \ .$$

(8)

Then relation (7) can be modified as follows

$$p_2 = a - bp_1 \ , \qquad p_3 = c + dp_1 \ .$$

(9)

Let us find the portfolio with required rate of return $\overline{r_p}$ and with minimum risk level at the same time. Using formula (1) for variance of three-asset portfolio and formulas for p_2 and p_3 (9) we have

$$\sigma_p^2 = p_1^2\sigma_1^2 + (a - bp_1)^2\sigma_2^2 + (c + dp_1)^2\sigma_3^2 + 2p_1(a - bp_1)k_{12} + 2p_1(c + dp_1)k_{13} + \\ + 2(a - bp_1)(c + dp_1)k_{23} \quad ,$$

from which

$$\sigma_p^{2'} = 2p_1\sigma_1^2 - 2(a-bp_1)b\sigma_2^2 + 2(c+dp_1)d\sigma_3^2 + 2(a-bp_1)k_{12} - 2p_1bk_{12} + 2(c+dp_1)k_{13} + \\ + 2p_1dk_{13} - 2b(c+dp_1)k_{23} + 2(a-bp_1)dk_{23} = 2p_1\left(\sigma_1^2 + b^2\sigma_2^2 + d^2\sigma_3^2 - 2bk_{12} + 2dk_{13} - 2dbk_{23}\right) - \\ - 2\left(ab\sigma_2^2 - cd\sigma_3^2 - ak_{12} - ck_{13} + bck_{23} - adk_{23}\right),$$

$$\sigma_p^{2''} = 2\left(\sigma_1^2 + b^2\sigma_2^2 + d^2\sigma_3^2 - 2bk_{12} + 2dk_{13} - 2dbk_{23}\right) = \\ = 2\left(\sigma_1^2 + b^2\sigma_2^2 + d^2\sigma_3^2 - 2b\rho_{12}\sigma_1\sigma_2 + 2d\rho_{13}\sigma_1\sigma_3 - 2db\rho_{23}\sigma_2\sigma_3\right) \geq 2(\sigma_1 - b\sigma_2 + d\sigma_3)^2 > 0 ,$$

as correlation coefficients have to always be within interval $\langle -1,1 \rangle$. From stated above, it is obvious that portfolio with weights

$$p_1 = \frac{ab\sigma_2^2 - cd\sigma_3^2 - ak_{12} - ck_{13} + (bc-ad) - k_{23}}{\sigma_1^2 + b^2\sigma_2^2 + d^3\sigma_3^2 - 2bk_{12} + 2dk_{13} - 2dbk_{23}} ,$$

$$p_2 = a - bp_1 ,$$

$$p_3 = c + dp_1 \qquad (10)$$

will be the one with required rate of return and the minimum risk level at the same time.

Example 2. Let us suppose that we have three assets with expected rates of return $\overline{r_1} = 10\%$, $\overline{r_2} = 14\%$, $\overline{r_3} = 18\%$, standard deviations $\sigma_1 = 12\%, \sigma_2 = 20\%, \sigma_3 = 28\%$ and covariances $k_{12} = 70$, $k_{13} = 80$, $k_{23} = 210$. Let us find the portfolio created from these assets which has rate of return 16%.

Solution. From formula (8) we have a = 0,5 , b = 2 , c = 0,5 , d = 1. Weights of our portfolio with required rate of return 16% can be calculated using formulas (10)

$$p_1 = \frac{400 - 392 - 35 - 43 + (1-0{,}5)210}{144 + 1600 + 784 - 280 + 172 - 840} = \frac{35}{1580} = 0{,}022 ,$$

$$p_2 = 0{,}5 - 0{,}044 = 0{,}456 ,$$

$$p_3 = 0{,}5 + 0{,}022 = 0{,}522 .$$

Portfolio with weights of $p_1 = 0{,}022$, $p_2 = 0{,}456$ a $p_3 = 0{,}522$ has expected rate of return $\overline{r_p} = 0{,}022 \cdot 10 + 0{,}456 \cdot 14 + 0{,}522 \cdot 18 = 16\%$. Variance of portfolio calculated from formula (1) is $\sigma_p^2 = 0{,}0697 + 83{,}1744 + 213{,}6275 + 1{,}4045 + 1{,}9753 + 99{,}9734 = 400{,}2248$ and standard deviation is $\sigma_p \doteq 20$.

CONCLUSION

From stated above, we can obtain the algorithm of diversification to three assets consists as follows:

1. For diversification, we use only assets that meet the condition $k_{ij} \leq \min\{\sigma_i, \sigma_j\}$ is met. If so, we order assets so that $k_{12} \leq k_{13} \leq k_{23}$.
2. Find the portfolio with the lowest standard deviation, i. e. risk level, while weights of assets in the portfolio can be calculated from formulas (6) or solving the system of equations (3). Then calculate expected rate of return and standard deviation of this portfolio.
3. Portfolio with lower expected rate of return does not have any practical sense.
4. If we require higher rate of return, then we create the portfolio with weights calculated using formulas (10). The level of risk will be higher as well.

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